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Aequat. Math. 83 (2012), 75–86

0001-9054/12/010075-12
published online December 24, 2011
DOI 10.1007/s00010-011-0106-x

Aequationes Mathematicae

Some generalization of Cauchy's and the quadratic functional equations

Radosław Łukasik

Abstract. We find the solutions $f, g, h\colon S \to H$ of each of the functional equations

$$\sum_{\lambda\in\Lambda} f(x+\lambda y) = |\Lambda| g(x) + h(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, Λ is a finite subgroup of the automorphism group of S, $(H, +)$ is an abelian group.

Mathematics Subject Classification (2010). 39B52.

Keywords. Cauchy functional equation, quadratic functional equation.

1. Introduction

The generalization of the quadratic functional equation

$$f(x+y) + f(x+\sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

where σ is an automorphism of the abelian group G such that $\sigma^2 = id_G$, f, $g\colon G \to \mathbb{C}$ was investigated by Stetkær [3].

In Stetkær [4] solved the functional equation

$$\frac{1}{N}\sum_{n=0}^{N-1} f(z+\omega^n \zeta) = g(z) + h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \ldots\}$ and ω is a primitive Nth root of unity, $f, g, h\colon \mathbb{C} \to \mathbb{C}$ are continuous.

In the present paper we give the complete solution of the following functional equation

$$\sum_{\lambda\in\Lambda} f(x+\lambda y) = Lg(x) + h(y), \quad x, y \in S,$$

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where $(S,+)$ is an abelian semigroup, Λ is a finite subgroup of the automorphism group of $S, L = card\Lambda, (H,+)$ is an abelian group uniquely divisible by $(L+1)!, f, g, h\colon S \to H$.

2. Main result

In this work we use a theorem and a corollary proved by Mazur and Orlicz [2], and generalized by Djoković [1]:

Theorem 1. *Let $(S,+)$ be an abelian semigroup, $n \in \mathbb{N}, (H,+)$ be an abelian group uniquely divisible by $(n+1)!, f\colon S \to H$ satisfies the equation*

$$\Delta_v^{n+1} f(u) = 0, \quad u, v \in S.$$

Then there exist $A_0 \in H$ and k-additive, symmetric mappings $A_k\colon S \to H$, $k \in \{1, \dots, n\}$ such that

$$f(x) = A_0 + A_1(x) + \cdots + A_n(x, \dots, x), \quad x \in S.$$

Corollary 1. *Let $(S,+)$ be an abelian semigroup, $n \in \mathbb{N}, (H,+)$ be an abelian group uniquely divisible by $(n+1)!, f, g\colon S \to H$ satisfy the equation*

$$\Delta_v^{n} f(u) = g(v), \quad u, v \in S.$$

Then there exist $A_0 \in H$ and k-additive, symmetric mappings $A_k\colon S \to H$, $k \in \{1, \dots, n\}$ such that

$$f(x) = A_0 + A_1(x) + \cdots + A_n(x, \dots, x), \quad x \in S,$$
$$g(x) = n! A_n(x, \dots, x), \quad x \in S.$$

We start with some lemmas.

Lemma 1. *For all $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^i = 0, \quad i \in \{1, \dots, n-1\}, \quad n \neq 1, \tag{1}$$

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^n = n!. \tag{2}$$

Proof. First we prove (1) by induction on n.

For $n = 2$ we have

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^i = -2 \cdot 1 + 1 \cdot 2 = 0.$$

Assume that (1) holds for $2, \dots, n$. Since

$$k \binom{n+1}{k} = (n+1) \binom{n}{k-1}, \quad k \in \{1, \dots, n\},$$

we have

$$\begin{aligned}\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}k &= (n+1)\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{k-1}\\ &= (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} = 0.\end{aligned}$$

Hence, for $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}&\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}k^{i+1} = (n+1)\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{k-1}k^{i}\\ &\quad= (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k+1)^{i} = (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{m=0}^{i}\binom{i}{m}k^{m}\\ &\quad= (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} + (n+1)\sum_{m=1}^{i}\binom{i}{m}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{m}\\ &\quad= (n+1)\sum_{m=1}^{i}\binom{i}{m}\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^{m} = 0.\end{aligned}$$

Now we prove (2) by induction on n.

For $n = 1$ we have

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^{n} = 1.$$

Assume that (2) holds for some $n \in \mathbb{N}$. Using (1) we get

$$\begin{aligned}&\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}k^{n+1} = (n+1)\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{k-1}k^{n}\\ &\quad= (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k+1)^{n} = (n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{m=0}^{n}\binom{n}{m}k^{m}\\ &\quad= (n+1)\sum_{m=0}^{n}\binom{n}{m}\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^{m} = (n+1)\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^{n}\\ &\quad= (n+1)!.\end{aligned}$$

$\square$

Lemma 2. *Let $(H,+)$ be an abelian group uniquely divisible by $n!$, $x_1,\dots,x_n\in H$ be such that*

$$\sum_{i=1}^{n} k^i x_i = 0, \quad k\in\{1,\dots,n\}.$$

Then $x_1=\dots=x_n=0$.

Proof. First, we notice that for the group $(H,+)$ from the unique divisibility by $n!$ we obtain the unique divisibility by $k!$ for each $k\in\{1,\dots,n\}$.

We prove this theorem by induction on n.

For $n=1$ we have $x_1=0$. Assume that the theorem holds for $n-1, n>1$. In view of Lemma 1 we have

$$0=\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\sum_{i=1}^{n}k^i x_i=\sum_{i=1}^{n}\left[\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^i\right]x_i=n!x_n.$$

Hence $x_n=0$ and

$$\sum_{k=1}^{n-1}k^i x_i=0.$$

By the induction hypothesis we obtain

$$x_1=\dots=x_{n-1}=0=x_n,$$

which finishes the proof. □

Lemma 3. *Let $(S,+)$ be an abelian semigroup, Λ be a finite subgroup of the automorphism group of $S, L:=card\Lambda, (H,+)$ be an abelian group uniquely divisible by $L!$. Further, let for each $k\in\{1,\dots,L\}A_k\colon S\to H$ be k-additive and symmetric mappings, $f\colon S\to H$ be a function given by the formula*

$$f(x)=A_1(x)+\dots+A_L(x,\dots,x), \quad x\in S.$$

Then a function f satisfies the equation

$$\sum_{\lambda\in\Lambda}f(x+\lambda y)=Lf(x)+\sum_{\lambda\in\Lambda}f(\lambda y), \quad x,y\in S \tag{3}$$

if and only if

$$\binom{k}{i}\sum_{\lambda\in\Lambda}A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i})=0,\ x,y\in S,\ 1\le i\le k-1,\ 2\le k\le L.$$

Proof. First we show that for $x,y\in S$

$$\sum_{\lambda\in\Lambda}f(x+\lambda y)-\sum_{\lambda\in\Lambda}f(\lambda y)-Lf(x)=\sum_{\lambda\in\Lambda}\sum_{i=1}^{L-1}\sum_{k=i+1}^{L}\binom{k}{i}A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}).$$

Indeed, we have the following sequence of identities

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} f(x+\lambda y) - \sum_{\lambda\in\Lambda} f(\lambda y) - Lf(x) \\
&\quad= \sum_{\lambda\in\Lambda}\sum_{k=1}^{L}[A_k(x+\lambda y,\dots,x+\lambda y) - A_k(\lambda y,\dots,\lambda y) - A_k(x,\dots,x)] \\
&\quad= \sum_{\lambda\in\Lambda}\sum_{k=2}^{L}\sum_{i=1}^{k-1}\binom{k}{i} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) \\
&\quad= \sum_{\lambda\in\Lambda}\sum_{i=1}^{L-1}\sum_{k=i+1}^{L}\binom{k}{i} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}), \quad x,y\in S.
\end{aligned}$$

By the above equality we obtain that if

$$\binom{k}{i}\sum_{\lambda\in\Lambda} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y\in S,\ 1,\le i\le k-1,\ 1\le k\le L,$$

then f satisfies (3).

Now we assume that f satisfies (3). Then

$$\sum_{\lambda\in\Lambda}\sum_{i=1}^{L-1}\sum_{k=i+1}^{L}\binom{k}{i} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0.$$

For each $i\in\{1,\dots,L-1\}$ we define $g_i\colon S\times S\to H$ by the formula

$$g_i(x,y) := \sum_{\lambda\in\Lambda}\sum_{k=i+1}^{L}\binom{k}{i} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}), \quad x,y\in S.$$

Since

$$g_i(x,my) = m^i g_i(x,y), \quad m\in\mathbb{N},\ x,y\in S,\ 1\le i\le L-1,$$

then

$$\sum_{i=1}^{L-1} m^i g_i(x,y) = \sum_{i=1}^{L-1} g_i(x,my) = 0, \quad m\in\mathbb{N},\ x,y\in S.$$

In view of Lemma 2 we obtain

$$g_i(x,y) = 0, \quad x,y\in S,\ i\in\{1,\dots,L-1\}.$$

Now, for each $i\in\{1,\dots,L-1\}, j\in\{1,\dots,L-i\}$, we define $h_{i,j}\colon S\times S\to H$ by the formula

$$h_{i,j} := \binom{i+j}{i}\sum_{\lambda\in\Lambda} A_{i+j}(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}), \quad x,y\in S.$$

Since

$$h_{i,j}(mx,y)=m^j h_{i,j}(x,y),\ m\in\mathbb{N},\ x,y\in S,\ 1\le j\le L-i,\ 1\le i\le L-1,$$

we have

$$\sum_{j=1}^{L-i} m^j h_{i,j}(x,y)=\sum_{j=1}^{L-i} h_{i.j}(mx,y)=g_i(mx,y)=0,$$

for $m\in\mathbb{N},\ x,y\in S,\ 1\le i\le L-1$. In view of lemma 2 we obtain

$$h_{i,j}(x,y)=0,\quad x,y\in S,\ 1\le j\le L-i,\ 1\le i\le L-1,$$

so we get

$$\binom{k}{i}\sum_{\lambda\in\Lambda}A_k(x,\ldots,x,\underbrace{\lambda y,\ldots,\lambda y}_{i})=0,\quad x,y\in S,\ 1\le i\le k-1,\ 2\le k\le L.$$

□

Theorem 2. *Let $(S,+)$ be an abelian semigroup, Λ be a finite subgroup of the automorphism group of $S, L:=card\Lambda, (H,+)$ be an abelian group. Further, let $f\colon S\to H$ satisfy Eq. (3), $g\colon S\to H$ be a function given by*

$$g(x):=-\sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}f\left(\sum_{\mu\in\Lambda_{i,j}}\mu x\right),\quad x\in S,$$

where $\Lambda_{i,j}\subset\Lambda$ are pairwise different sets such that $card\Lambda_{i,j}=L-i$ for $j\in\{1,\ldots,\binom{L}{i}\}, i\in\{0,\ldots,L\}$. Then

$$L\Delta_v^L f(u)=Lg(v),\quad u,v\in S.$$

Proof. First we observe that any solution of Eq. (3) in a semigroup without zero can be uniquely extended to a solution in a semigroup with a neutral element 0 by putting $f(0)=0$. So we can assume that $(S,+)$ is a monoid.

Since

$$\lambda\Lambda_{i,j}=\Lambda_{i,k}\Rightarrow\lambda^{-1}\Lambda_{i,k}=\Lambda_{i,j},\quad\lambda\in\Lambda, i\in\{0,\ldots,L\}, j,k\in\left\{1,\ldots,\binom{L}{i}\right\},$$

we have

$$\sum_{j=1}^{\binom{L}{i}}\sum_{\lambda\in\Lambda}f\left(\sum_{\mu\in\Lambda_{i,j}}\lambda\mu x\right)=L\sum_{j=1}^{\binom{L}{i}}f\left(\sum_{\mu\in\Lambda_{i,j}}\mu x\right),\quad x\in S.\tag{4}$$

Fix $u,v\in S$. Let

$$x_i:=u+iv,\quad y_{i,j}:=\sum_{\mu\in\Lambda_{i,j}}\mu v,\quad j\in\left\{1,\ldots,\binom{L}{i}\right\},\quad i\in\{0,\ldots,L\}.$$

For $\lambda \in \Lambda, i \in \{0,\dots,L\}, j \in \{1,\dots,\binom{L}{i}\}$ we consider two cases:

(i) $\lambda^{-1} \in \Lambda_{i,j}$. Hence $i \neq L$. Let $k \in \{1,\dots,\binom{L}{i+1}\}$ be such that $\Lambda_{i,j} = \Lambda_{i+1,k} \cup \{\lambda^{-1}\}$. Then

$$\begin{aligned} x_i + \lambda y_{i,j} &= u + iv + \sum_{\mu\in\Lambda_{i,j}} \lambda\mu v = u + (i+1)v + \sum_{\mu\in\Lambda_{i+1,k}} \lambda\mu v \\ &= x_{i+1} + \lambda y_{i+1,k}. \end{aligned}$$

(ii) $\lambda^{-1} \notin \Lambda_{i,j}$. Hence $i \neq 0$. Let $k \in \{1,\dots,\binom{L}{i-1}\}$ be such that $\Lambda_{i-1,k} = \Lambda_{i,j} \cup \{\lambda^{-1}\}$. Then

$$\begin{aligned} x_i + \lambda y_{i,j} &= u + iv + \sum_{\mu\in\Lambda_{i,j}} \lambda\mu v = u + (i-1)v + \sum_{\mu\in\Lambda_{i-1,k}} \lambda\mu v \\ &= x_{i-1} + \lambda y_{i-1,k}. \end{aligned}$$

From the above consideration and equality (4) we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^{L}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}\sum_{\lambda\in\Lambda} f(x_i + \lambda y_{i,j}) \\ &= \sum_{i=0}^{L}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}\left[Lf(x_i) + \sum_{\lambda\in\Lambda} f(\lambda y_{i,j})\right] \\ &= L\sum_{i=0}^{L}(-1)^{L-i}\binom{L}{i} f(u+iv) + \sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}\sum_{\lambda\in\Lambda} f\left(\sum_{\mu\in\Lambda_{i,j}}\lambda\mu v\right) \\ &= L\sum_{i=0}^{L}(-1)^{L-i}\binom{L}{i} f(u+iv) + L\sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}} f\left(\sum_{\mu\in\Lambda_{i,j}}\lambda\mu v\right) \\ &= L\Delta_v^L f(u) - Lg(v). \end{aligned}$$

□

Theorem 3. *Let $(S,+)$ be an abelian semigroup, Λ be a finite subgroup of the automorphism group of $S, L := card\Lambda, (H,+)$ be an abelian group uniquely divisible by $(L+1)!$. Then a function $f\colon S \to H$ satisfies Eq. (3) if and only if there exist k-additive and symmetric mappings $A_k\colon S^k \to H, k \in \{1,\dots L\}$ such that*

$$f(x) = A_1(x) + \cdots + A_L(x, \ldots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \ldots, x, \underbrace{\lambda y, \ldots, \lambda y}_{i}) = 0, \quad x, y \in S,\ 1 \le i \le k-1,\ 2 \le k \le L.$$

Proof. The proof of this theorem follows straight from Theorem 2, Corollary 1 and Lemma 3. □

Now we show some applications of the above theorems.

Theorem 4. *Let $(S, +)$ be an abelian semigroup, Λ be a finite subgroup of the automorphism group of $S, L := card\Lambda, (H, +)$ be an abelian group uniquely divisible by $(L+1)!$. Then a function $f\colon S \to H$ satisfies the equation*

$$\sum_{\lambda \in \Lambda} f(x + \lambda y) = Lf(x) + Lf(y), \quad x, y \in S \tag{5}$$

if and only if there exist k-additive, symmetric mappings $A_k\colon S^k \to H$, $k \in \{1, \ldots L\}$ such that

$$f(x) = A_1(x) + \cdots + A_L(x, \ldots, x), \quad x \in S,$$

$$\binom{k}{i} \sum_{\lambda \in \Lambda} A_k(x, \ldots, x, \underbrace{\lambda y, \ldots, \lambda y}_{i}) = 0, \quad x, y \in S, \quad 1 \le i \le k-1,\ 2 \le k \le L,$$

$$A_k(\mu x, \ldots, \mu x) = A_k(x, \ldots, x), \quad x \in S,\ \mu \in \Lambda,\ 1 \le k \le L.$$

Proof. It is easy to check that if f satisfies the three above conditions then f satisfies Eq. (5). We observe that if f satisfies Eq. (5), then f satisfies Eq. 3 and $f(\mu x) = f(x),\ \mu \in \Lambda,\ x \in S$.

Indeed, for $x, y \in S, \mu \in \Lambda$ we have

$$Lf(y) = -Lf(x) + \sum_{\lambda \in \Lambda} f(x + \lambda y) = -Lf(x) + \sum_{\lambda \in \Lambda} f(x + \lambda \mu y) = Lf(\mu y),$$

which gives $f(\mu x) = f(x)$. Hence $Lf(y) = \sum_{\lambda \in \Lambda} f(\lambda y)$ for $y \in S$, thus f satisfies Eq. (3). In view of Theorem 3 we obtain the first and the second condition of the thesis. Finally, we show that

$$A_k(\mu x, \ldots, \mu x) = A_k(x, \ldots, x), \quad x \in S, \quad \mu \in \Lambda, \quad 1 \le k \le L.$$

Fix $\mu \in \Lambda$. We observe that

$$0 = f(\mu x) - f(x) = \sum_{k=1}^{L} [A_k(\mu x, \ldots, \mu x) - A_k(x, \ldots, x)], \quad x \in S.$$

For each $i \in \{1, \ldots, L\}$ we define $g_i\colon S \to H$ by the formula

$$g_i(x) := A_i(\mu x, \ldots, \mu x) - A_i(x, \ldots, x), \quad x \in S.$$

Since

$$g_i(mx) = m^i g(x), \quad m \in \mathbb{N}, \quad x \in S,$$

we have

$$0=\sum_{i=1}^{L} g_i(mx)=\sum_{i=1}^{L} m^i g_i(x), \quad m\in\mathbb{N}, \quad x\in S.$$

Hence, in view of Lemma 2, we obtain

$$g_i(x)=0, \quad 1\le i\le L, \quad x\in S,$$

which ends the proof. $\square$

Theorem 5. *Let $(S,+)$ be an abelian semigroup, Λ be a finite subgroup of the automorphism group of $S, L:=card\Lambda, (H,+)$ be an abelian group uniquely divisible by $L!$. Then a function $f\colon S\to H$ satisfies the equation*

$$\sum_{\lambda\in\Lambda} f(x+\lambda y)=Lf(x), \quad x,y\in S \tag{6}$$

if and only if there exist k-additive, symmetric mappings $A_k\colon S^k\to H$, $k\in\{1,\dots L-1\}$ and $A_0\in H$ such that

$$f(x)=A_0+A_1(x)+\cdots+A_{L-1}(x,\dots,x), \quad x\in S,$$
$$\binom{k}{i}\sum_{\lambda\in\Lambda} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i})=0, \quad x,y\in S,\ 1\le i\le k, \quad 2\le k\le L-1.$$

Proof. Let us note that if a function f satisfies the above conditions then f satisfies Eq. (6).

First we observe that if $(S,+)$ is a semigroup without a neutral element 0, then we can uniquely extend each solution of (6) to a solution on the semigroup $(S\cup\{0\},+)$. Indeed, we have

$$\begin{aligned}L\sum_{\lambda\in\Lambda} f(\lambda y)&=\sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} f(\lambda y+\mu x)\\&=\sum_{\mu\in\Lambda}\sum_{\lambda\in\Lambda} f(\mu x+\lambda y)=L\sum_{\mu\in\Lambda} f(\mu x), \quad x,y\in S,\end{aligned}$$

and

$$Lf\left(\sum_{\lambda\in\Lambda}\lambda y\right)=\sum_{\mu\in\Lambda} f\left(\mu\sum_{\lambda\in\Lambda}\lambda y\right)=\sum_{\mu\in\Lambda} f(\mu x), \quad x,y\in S.$$

Putting $f(0):=f(\sum_{\lambda\in\Lambda}\lambda y)$ for $y\in S$. It is easy to check that the solution extended in this way satisfies Eq. 3 for the semigroup S with zero.
So we can assume that $(S,+)$ is a monoid. Since

$$Lf(0)=\sum_{\lambda\in\Lambda} f(\lambda y), \quad y\in S,$$

then $f_0 := f - f(0)$ satisfies Eq. 3. Let $\Lambda_{i,j} \subset \Lambda, j \in \{1, \dots, \binom{L}{i}\}, i \in \{0, \dots, L\}$, be sets from Theorem 2. We observe that for $x \in S$ we have

$$\sum_{\lambda\in\Lambda} f_0\left(\sum_{\mu\in\Lambda_{i,j}} \lambda\mu x\right) = \sum_{\lambda\in\Lambda} f\left(\lambda \sum_{\mu\in\Lambda_{i,j}} \mu x\right) - Lf(0) = 0.$$

By equality (4) we obtain

$$L\sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}} f_0\left(\sum_{\mu\in\Lambda_{i,j}} \mu x\right) = \sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}\sum_{\lambda\in\Lambda} f_0\left(\sum_{\mu\in\Lambda_{i,j}} \lambda\mu x\right) = 0.$$

Then Theorem 2 holds with $g = 0$. In view of Theorem 1, Lemma 3 and putting $A_0 := f(0)$ we obtain that there exist k-additive and symmetric mappings $A_k : S^k \to H, k \in \{1, \dots L-1\}$ such that

$$f(x) = A_0 + A_1(x) + \cdots + A_{L-1}(x, \dots, x), \quad x \in S,$$

$$\binom{k}{i}\sum_{\lambda\in\Lambda} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_{i}) = 0, \quad x, y \in S, \quad 1 \le i \le k-1, \quad 2 \le k \le L-1,$$

Finally, we show that

$$\sum_{\lambda\in\Lambda} A_k(\lambda x, \dots, \lambda x) = 0, \quad x \in S, \quad 2 \le k \le L-1.$$

For each $i \in \{1, \dots, L-1\}$ we define $g_i : S \to H$ by the formula

$$g_i(x) := \sum_{\lambda\in\Lambda} A_i(\lambda x, \dots, \lambda x), \quad x \in S.$$

Since

$$g_i(mx) = m^i g(x), m \in \mathbb{N}, \quad x \in S, \quad 1 \le i \le L-1,$$

we have

$$0 = \sum_{\lambda\in\Lambda} f(\lambda x) - Lf(0) = \sum_{i=1}^{L} g_i(mx) = \sum_{i=1}^{L} m^i g_i(x), \quad m \in \mathbb{N}, \quad x \in S.$$

In view of Lemma 2

$$g_i(x) = 0, \quad 1 \le i \le L-1, \quad x \in S,$$

which ends the proof. □

Our main result reads as follows

Theorem 6. *Let $(S, +)$ be an abelian monoid, Λ be a finite subgroup of the automorphism group of $S, L := card\Lambda, (H, +)$ be an abelian group uniquely divisible by $(L+1)!$. Then functions $f, g, h : S \to H$ satisfy the equation*

$$\sum_{\lambda\in\Lambda} f(x + \lambda y) = Lg(x) + h(y), \quad x, y \in S \tag{7}$$

if and only if there exist k-additive, symmetric mappings $A_k\colon S^k \to H$, $k \in \{1,\dots L\}$ *and* $A_0, B_0 \in H$ *such that*

$$f(x) = A_0 + A_1(x) + \cdots + A_L(x,\dots,x), \quad x \in S,$$
$$g(x) = B_0 + A_1(x) + \cdots + A_L(x,\dots,x), \quad x \in S,$$
$$h(x) = LA_0 - LB_0 + \sum_{\lambda\in\Lambda} A_1(\lambda x) + \cdots + \sum_{\lambda\in\Lambda} A_L(\lambda x,\dots,\lambda x), \quad x \in S,$$
$$\binom{k}{i} \sum_{\lambda\in\Lambda} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y \in S, \quad 1 \le i \le k-1, \ 2 \le k \le L.$$

Proof. It is easy to check that if the four equalities above hold, then functions f, g, h satisfy Eq. (7).

Assume that functions f, g, h satisfy Eq. (7). First we observe that

$$Lf(0) = Lg(0) + h(0),$$
$$Lf(x) = Lg(x) + h(0) = Lg(x) - Lg(0) + Lf(0), \quad x \in S.$$

Therefore, we get

$$f(x) = g(x) - g(0) + f(0), \quad x \in S, \tag{8}$$

and

$$Lg(0) + h(y) = \sum_{\lambda\in\Lambda} f(\lambda y) = \sum_{\lambda\in\Lambda} [g(\lambda y) - g(0)] + Lf(0), \quad y \in S,$$

hence

$$h(y) = \sum_{\lambda\in\Lambda} [g(\lambda y) - g(0)] + Lf(0) - Lg(0), \quad y \in S. \tag{9}$$

Let $g_0 := g - g(0)$. From the above equalities we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} g_0(x+\lambda y) &= \sum_{\lambda\in\Lambda} [g(x+\lambda y) - g(0) + f(0)] - Lf(0) \\ &= \sum_{\lambda\in\Lambda} f(x+\lambda y) - Lf(0) = Lg(x) + h(y) - Lf(0) \\ &= Lg_0(x) + \sum_{\lambda\in\Lambda} g_0(\lambda y), \quad x,y \in S.\end{aligned}$$

In view of Theorem 3, there exist k-additive and symmetric mappings $A_k\colon S^k \to H, k \in \{1,\dots L\}$ such that

$$g_0(x) = A_1(x) + \cdots + A_L(x,\dots,x), \quad x \in S,$$
$$\binom{k}{i} \sum_{\lambda\in\Lambda} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y \in S, \quad 1 \le i \le k-1, \quad 2 \le k \le L.$$

Let $B_0 := g(0), A_0 := f(0)$. Then from equalities 8 and 9 we obtain the thesis. □

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Received: November 12, 2010
Revised: November 9, 2011